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On Uniform Spline Approximation*

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Let Δ be a partition of [0, 1], m be an integer greater than two, and $\mathscr{S}$ be the set of spline functions of order m (degree $m - 1$) with knots in Δ. It is proved that for functions $f(x)$ continuous on [0, 1],

$$\operatorname{dist}(f, \mathscr{S}) \leqslant \left(\sqrt{\frac{m}{12}} + 1\right) \omega(|\Delta|),$$

where $\omega(\cdot)$ is the modulus of continuity of $f(x)$, $|\Delta|$ is the mesh of Δ, and distance is measured by the sup norm. The proof uses the variation diminishing spline approximation method developed by Schoenberg [13] to get a spline function whose distance from $f(x)$ is bounded by the given expression. Similar bounds on $\operatorname{dist}(f, \mathscr{S})$ have been obtained by de Boor [5], but his coefficients are much larger and are not easily computed for large m.

A similar estimate is given for Tchebycheffian spline functions.

1. Spline Functions

Let $m > 1$ and $n > 0$ be integers and $\Delta = \{x_i\}_0^n$ be a finite sequence of real numbers satisfying

$$0 = x_0 < x_1 \leqslant x_2 \leqslant \cdots \leqslant x_{n-1} < x_n = 1$$

and

$$x_{i-m+1} < x_i \qquad \text{for} \quad m \leqslant i < n.$$

By a spline function [4, 13, 14] of order m, or degree $m - 1$, is meant a real function $s(x)$ defined on [0, 1] such that

(i) $s(x)$ is a polynomial of degree $m - 1$ or less on $(x_j, x_{j+1}]$ for $0 \leqslant j \leqslant n - 1$, whenever $x_j < x_{j+1}$; and

(ii) $s(x)$ has a continuous $(m - i)$-th derivative on (x_j, x_{j+i}) for $0 \leqslant j \leqslant n - 2$, whenever $x_j < x_{j+i}$.

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Of course, the latter condition conveys no additional information unless one or more knots of Δ are interior to (x_j, x_{j+i}).

2. Concerning Bounds on $|Sf(x) - f(x)|$

Schoenberg [13] has devised an approximation method that associates to each $f(x)$ defined in [0, 1] a spline function $Sf(x)$ of order m with knots in Δ by the formula

$$Sf(x) = \sum f(\xi_j)\, N_j(x), \tag{2.1}$$

where ξ_j and $N_j(x)$ depend on m and Δ but not on $f(x)$. Here we shall require the following facts:

$$N_j(x) \geqslant 0 \text{ for each } j; \tag{2.2}$$

$$\sum N_j(x) = 1 \text{ on } [0, 1]; \tag{2.3}$$

$$\sum \xi_j N_j(x) = x \text{ on } [0, 1]; \tag{2.4}$$

and for the function $g(x) = x^2$, the error

$$E(x) = Sg(x) - g(x) \tag{2.5}$$

satisfies [9]

$$0 \leqslant E(x) = \sum \xi_j^2 N_j(x) - x^2 \leqslant h^2(m, \Delta), \tag{2.6}$$

where

$$h(m, \Delta) = \min \left\{ \frac{1}{\sqrt{2m-2}}, \sqrt{\frac{m}{12}} \mid \Delta \mid \right\}. \tag{2.7}$$

The proof of (2.6) is similar to the proof of (4.1) in [9] except that the bound $x_s - x_r \leqslant (s - r) \mid \Delta \mid$ is used instead of the looser bound $x_s - x_r \leqslant k \mid \Delta \mid$. See also (6.1) in [10].

For a fixed continuous $f(x)$ with modulus of continuity $\omega(\cdot)$,

$$\begin{aligned} \mid Sf(x) - f(x) \mid &\leqslant \sum \mid f(\xi_j) - f(x) \mid N_j(x) \\ &\leqslant \sum \omega(\mid \xi_j - x \mid)\, N_j(x) \\ &= \sum \omega(\mid \xi_j - x \mid \delta^{-1}\delta)\, N_j(x) \\ &\leqslant \sum (\mid \xi_j - x \mid \delta^{-1} + 1)\, \omega(\delta)\, N_j(x) \\ &= \omega(\delta) \left(\delta^{-1} \sum \mid \xi_j - x \mid N_j(x) + 1 \right), \end{aligned}$$

where $\delta > 0$. On the other hand,

$$\begin{aligned}\left(\sum |\xi_j - x| N_j(x)\right)^2 &= \left(\sum |\xi_j - x| \sqrt{N_j(x)} \cdot \sqrt{N_j(x)}\right)^2 \\ &\leqslant \sum (\xi_j - x)^2 N_j(x) \cdot \sum N_j(x) \\ &= \sum (\xi_j - x)^2 N_j(x) \\ &= \sum \xi_j^2 N_j(x) - 2x \sum \xi_j N_j(x) + x^2 \sum N_j(x) \\ &= \sum \xi_j^2 N_j(x) - x^2 = E(x)\end{aligned}$$

in view of (2.2–4). Thus

$$| Sf(x) - f(x) | \leqslant \omega(\delta) (\delta^{-1}\sqrt{E(x)} + 1), \tag{2.8}$$

and if we define

$$\rho(f, g) = \max | f(x) - g(x) |,$$

we have

$$\rho(Sf, f) \leqslant \omega(\delta)(\delta^{-1}h(m, \Delta) + 1). \tag{2.9}$$

Choosing $\delta = | \Delta |$ gives

$$\rho(Sf, f) \leqslant \omega(| \Delta |) (1 + \sqrt{m/12}). \tag{2.10}$$

Choosing $\delta = 1/\sqrt{m-1}$ gives

$$\rho(Sf, f) \leqslant \omega(1/\sqrt{m-1})(1 + 1/\sqrt{2}). \tag{2.11}$$

3. Concerning dist(f, $\mathscr{S}$)

For $\mathscr{S}$ = the set of all spline functions of order m over Δ, we let

$$\operatorname{dist}(f, \mathscr{S}) = \inf_{s \in \mathscr{S}} \rho(s, f).$$

Since $Sf(x)$ is in $\mathscr{S}$, (2.10) and (2.11) give bounds on dist(a, $\mathscr{S}$). However, since the polynomials of degree $m - 1$ are also in $\mathscr{S}$, (2.11) conveys no new information because of Popoviciu's [12] estimate on the error in Bernstein polynomial approximation.

For various m we now denote $\mathscr{S}$ by $\mathscr{S}^{m-1}$ and list our results for $m = 3, 4, 5, 6$:

$$\begin{aligned}\operatorname{dist}(f, \mathscr{S}^2) &\leqslant 1.50\, \omega(| \Delta |) \\ \operatorname{dist}(f, \mathscr{S}^3) &\leqslant 1.58\, \omega(| \Delta |) \\ \operatorname{dist}(f, \mathscr{S}^4) &\leqslant 1.65\, \omega(| \Delta |) \\ \operatorname{dist}(f, \mathscr{S}^5) &\leqslant 1.71\, \omega(| \Delta |)\end{aligned} \tag{3.1}$$

4. Remarks

a. The bound given by (2.10) is a liberal overestimate. Moreover, for convex functions at least, the error

$$Sf(x) - f(x)$$

is nonnegative and hence at least twice dist(f, $\mathscr{S}$).

b. Replacing $\omega(\cdot)$ by

$$\omega^*(\cdot) = \inf \omega(\cdot; f(x) - cx)$$

does not change the validity of (2.10) above. Observe that for $f(x) = \alpha + \beta x$, (2.10) with $\omega(\cdot)$ does not give zero while (2.10) with $\omega^*(\cdot)$ does. See [10].

c. While it may be possible to use $Sf(x)$ to prove a result similar to de Boor's Corollary 2 in [5], a paraphrasing of the proof there will not work because S is not a projection operator.

d. The derivation of (2.8), follows that used to arrive at Popoviciu's estimate. See [10–12].

e. Shisha and Mond [15] have recently considered bounds on approximation by linear positive operators in general. Using their result would precisely double the right sides of (2.10), (2.11) and (3.1) above. This is because their results were derived independently of an assumption like (2.4) above. Linear positive operators have also been considered by Bohman [1] and Korovkin [8].

f. For Tchebycheffian spline functions (see [6, 7, 9]), it is convenient to use Theorem 2 of (15) with

$$\begin{aligned} F(t, x) &= \int_x^t w_1(y) \int_x^y w_2(z)\, dz\, dy \\ &\geqslant 1/2\, (\min w_1(y))\, (\min w_2(y))\, (t - x)^2 \end{aligned}$$

on the linear positive operator T developed in [9] (see also [6]). Then

$$E(x) = Tu_2(x) - u_2(x) = \sum F(\xi_j, x)\, N_j(x).$$

A slight modification of the proof of (9.7) in [9] gives the improved result

$$0 \leqslant E(x) \leqslant 1/2\, (\max w_1(Y))(\max w_2(Y))(m - 1)^2 \mid \Delta \mid^2$$

whence

$$\begin{aligned} \mid Tf(x) - f(x) \mid &\leqslant 2\omega(C(m - 1) \mid \Delta \mid) \\ &\leqslant (2C(m - 1) + 2)\, \omega(\mid \Delta \mid), \end{aligned}$$

where

$$C = (\max w_1(Y))\, (\max w_2(Y))\, (\min w_1(Y))^{-1}\, (\min w_2(Y))^{-1}.$$

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